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ABSTRACT

This paper describes a simple class of homogeneous, isotropic, *compressible* hyperelastic materials capable of sustaining nontrivial states of finite anti-plane shear.

1. Introduction. One of the simplest classes of deformations of solids is that of anti-plane shear, in which each particle of a cylinder is displaced axially by an amount that depends on the position of the particle in its cross-section but not on the axial coordinate of the particle. In the linearized theory of *infinitesimal* deformations, every homogeneous, isotropic elastic material is capable of sustaining, in the absence of body force, anti-plane shears which are nontrivial in the sense that they are not *simple* shears. When *finite* deformations are considered, however, this is not the case. In the corresponding nonlinear theory, anti-plane shear is accompanied by normal stresses, and as a result the local equations of equilibrium may be violated unless suitably contrived body forces are present.

Maximal subclasses of homogeneous, isotropic hyperelastic materials capable of sustaining nontrivial finite anti-plane shear were determined in [1] for the case of incompressible materials and in [2] for compressible ones. The class of incompressible

materials delineated in [1] includes an especially simple subclass that has proved to be useful in studying nonlinear effects - especially qualitative ones - in a simple setting; see, for example, [3,4,5]. The materials in this subclass have been termed *generalized Neo-Hookean* materials by Gurtin [4] because the Neo-Hookean material, sometimes used as a simple model for the mechanical behavior of rubber, is included among them.

The class of *compressible* homogeneous, isotropic hyperelastic materials that sustain nontrivial anti-plane shear is more restricted than its incompressible counterpart. Our purpose here is to set out a special class of elastic potentials for compressible materials for which nontrivial anti-plane shear is not only possible, but is described by field equations comparable in simplicity to those for the generalized Neo-Hookean materials in the incompressible case.

In the following section, we derive the field equations applicable to finite anti-plane shear for a homogeneous, isotropic compressible hyperelastic material. In Section 3, we specify the special subclass of materials to be studied, and we show that nontrivial anti-plane shear is possible for all materials in this class. Section 4 is concerned with a particular material that is the analog, within the special class considered, of the Neo-Hookean incompressible material.

2. Finite anti-plane shear. Consider a homogeneous, isotropic hyperelastic body that occupies a cylindrical region R in the reference state. Let x_1, x_2, x_3 be rectangular cartesian coordinates with the x_3- axis parallel to the generators of the cylinder. A deformation is an *anti-plane shear* if it carries the particle with coordinates x_i to the point with coordinates y_i, with

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + u(x_1, x_2), \tag{1}$$

where the out-of-plane displacement u is a twice continuously differentiable function on the open cross-section D of R. In the given cartesian frame, the components of the deformation gradient tensor $\underset{\sim}{F}$ are given by

$$\left.\begin{aligned} F_{\alpha\beta} = y_{\alpha,\beta} = \delta_{\alpha\beta}, \qquad F_{\alpha 3} = y_{\alpha,3} = 0, \\ F_{3\alpha} = y_{3,\alpha} = u_{,\alpha}, \qquad F_{33} = y_{3,3} = 1. \end{aligned}\right\} \tag{2}$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta, Latin and Greek subscripts have the respective ranges 1,2,3 and 1,2 , and repeated subscripts are summed over the appropriate range. A subscript preceded by a comma indicates partial differentiation with respect to the corresponding x - coordinate. The left Cauchy - Green tensor $\underset{\sim}{G} = \underset{\sim}{F}\underset{\sim}{F}^T$ has components

$$G_{\alpha\beta} = G_{\beta\alpha} = \delta_{\alpha\beta}, \quad G_{3\alpha} = G_{\alpha 3} = u_{,\alpha}, \quad G_{33} = 1 + k^2, \tag{3}$$

where we have written $k = (u_{,\alpha} u_{,\alpha})^{1/2} = |\nabla u|^{1/2}$. The fundamental scalar invariants I_1, I_2, I_3 of $\underset{\sim}{G}$ are given by

$$I_1 \equiv \mathrm{Tr}\,\underset{\sim}{G} = 3 + k^2, \quad I_2 \equiv 1/2\left[(\mathrm{Tr}\,\underset{\sim}{G})^2 - \mathrm{Tr}(\underset{\sim}{G}^2)\right] = 3 + k^2, \quad I_3 \equiv \det(\underset{\sim}{G}^2) = 1. \tag{4}$$

The deformation (1) is called a *simple shear* if the gradient of u is constant on D; in this case, the constant $k = |\nabla u|$ is called the amount of shear.

Let $W(I_1, I_2, I_3)$ be the elastic potential for a homogeneous, isotropic material. In any deformation, the nominal stress tensor $\underset{\sim}{\sigma}$ is related to the deformation gradient tensor $\underset{\sim}{F}$

through

$$\underset{\sim}{\sigma} = 2\frac{\partial W}{\partial I_1}\underset{\sim}{F} + 2\frac{\partial W}{\partial I_2}(I_1\underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{F} + 2\, I_3\frac{\partial W}{\partial I_3}\underset{\sim}{F}^{-T} , \tag{5}$$

where $\underset{\sim}{G}$ is related to $\underset{\sim}{F}$ through (2), $\underset{\sim}{1}$ stands for the identity tensor, and $\underset{\sim}{F}^{-T}$ is the transpose of the inverse of the nonsingular tensor $\underset{\sim}{F}$. The true stress tensor $\underset{\sim}{\tau}$ is given in terms of $\underset{\sim}{\sigma}$ and $\underset{\sim}{F}$ by

$$\underset{\sim}{\tau} = \underset{\sim}{\sigma}\,\underset{\sim}{F}^{T}/J , \tag{6}$$

where $J = \det\underset{\sim}{F} = \sqrt{I_3} > 0$ is the Jacobian of the deformation. From (5) and (6), one has

$$\underset{\sim}{\tau} = (2/J)\left[\frac{\partial W}{\partial I_1}\underset{\sim}{G} + \frac{\partial W}{\partial I_2}(I_1\underset{\sim}{1} - \underset{\sim}{G})\underset{\sim}{G} + I_3\frac{\partial W}{\partial I_3}\underset{\sim}{1}\right] . \tag{7}$$

When specialized to the anti-plane shear (1), the constitutive statement (5) yields the components of $\underset{\sim}{\sigma}$ as

$$\left.\begin{aligned}
\sigma_{\alpha\beta} &= \left(2\frac{\partial W_o}{\partial I_1} + 2(2 + |\nabla u|^2)\frac{\partial W_o}{\partial I_2} - p\right)\delta_{\alpha\beta} - 2\frac{\partial W_o}{\partial I_2}u_{,\alpha}u_{,\beta} ,\\
\sigma_{\alpha 3} &= \left(-2\frac{\partial W_o}{\partial I_2} + p\right)u_{,\alpha} , \qquad \sigma_{3\alpha} = 2\left(\frac{\partial W_o}{\partial I_1} + \frac{\partial W_o}{\partial I_2}\right)u_{,\alpha} ,\\
\sigma_{33} &= 2\frac{\partial W_o}{\partial I_1} + 4\frac{\partial W_o}{\partial I_2} - p ,
\end{aligned}\right\} \tag{8}$$

where we have written

$$Wo = Wo(I_1, I_2) \equiv W(I_1, I_2, 1)\ , \qquad p = -2\frac{\partial Wo}{\partial I_3}(I_1, I_2, 1)\ , \tag{9}$$

and , from (4),

$$I_1 = I_2 = 3 + |\nabla u|^2\ . \tag{10}$$

In the absence of body forces, the equations of local equilibrium are

$$\sigma_{ij,j} = 0 \quad \text{on } R. \tag{11}$$

From (8), (11), one finds that equilibrium holds for the anti-plane shear (1) if and only if the out-of-plane displacement $u(x_1, x_2)$ satisfies all three of the following differential equations on D:

$$\left[p - 2\frac{\partial Wo}{\partial I_1} - 2(2+|\nabla u|^2)\frac{\partial Wo}{\partial I_2}\right]_{,\alpha} + \left[2\frac{\partial Wo}{\partial I_2}u_{,\alpha}u_{,\beta}\right]_{,\beta} = 0\ , \quad \alpha = 1,2\ , \tag{12}$$

$$\left[M(|\nabla u|)\, u_{,\beta}\right]_{,\beta} = 0\ , \tag{13}$$

where the arguments I_1 and I_2 of p and Wo in (12) are given by (10), and M in (13) is defined by

$$M(k) = 2\left[\frac{\partial Wo}{\partial I_1}(I_1, I_2) + \frac{\partial Wo}{\partial I_2}(I_1, I_2)\right]_{I_1 = I_2 = 3+k^2}\ . \tag{14}$$

Consider momentarily the special case of (1) for which $u = kx_2$, corresponding to a simple shear in which the constant k is the amount of shear. By specializing (3), (4) and (7) to this case, one finds that the associated shear stress is given by $\tau_{32} = M(k)k$; thus $M(k)$ may be interpreted as the secant modulus of shear in a simple shear of amount k. Although (12) and (13) are satisfied by $u = kx_2$ for *any* isotropic material, they comprise an over-determined system in general. We turn now to a special class of isotropic materials for which the two equations (12) are trivially satisfied for *any* u, so that the equilibrium requirements for finite anti-plane shear reduce to the single scalar equation (13) alone.

3. A special class of materials. For *incompressible* homogeneous, isotropic hyperelastic materials, the third invariant I_3 has the value unity for all deformations, and the elastic potential W depends only on I_1 and I_2. The subclass of these materials for which W is independent of I_2 comprises the generalized Neo-Hookean materials. We now assume that the compressible material to be treated here has the analogous property:

$$W(I_1, I_2, I_3) = W(I_1, I_3) \quad . \tag{15}$$

If W is independent of I_2, one finds with the help of (9) that $Wo = Wo\,(I_1) = W(I_1, 1)$, and the first two equilibrium equations (12) for anti-plane shear specialize to

$$\left[\frac{\partial W}{\partial I_1}(I_1, 1) + \frac{\partial W}{\partial I_3}(I_1, 1)\right]_{,\alpha} = 0 \ , \qquad \alpha = 1,2, \tag{16}$$

with $I_1 = 3 + |\nabla u|^2$. We now require that $W(I_1, I_3)$ be of the following special form:

$$W(I_1, I_3) = Wo\,(I_1) + Wo'\,(I_1)\, f(I_3) + g(I_3) \ , \tag{17}$$

where the prime indicates differentiation, and Wo , f and g are functions on the interval $(0,\infty)$, with Wo differentiable three times, f and g twice. Moreover, f and g are required to satisfy

$$f(1) = 0, \quad f'(1) = -1, \quad g(1) = 0\,. \tag{18}$$

From (17) and $(18)_{1,2}$, one has

$$\frac{\partial W}{\partial I_1}(I_1, 1) + \frac{\partial W}{\partial I_3}(I_1, 1) = g'(1) = \text{constant}, \tag{19}$$

so that both equations in (16) are satisfied for any choice of the out-of-plane displacement u. By setting u = 0 in (8) and making use of (9), (17) and $(18)_{1,2}$, one finds that the components of stress in the reference configuration are given by

$$\sigma_{ij} = 2\, g'(1)\, \delta_{ij} \;; \tag{20}$$

If the reference configuration is to be unstressed, as we assume from here on, then g must also satisfy

$$g'(1) = 0 \tag{21}$$

in addition to $(18)_3$. According to (14), the secant modulus M(k) now becomes

$$M(k) = 2W_o'\,(3 + k^2)\;, \tag{22}$$

so that the surviving equilibrium equation (13) may be written as

$$\left[2\, W_0'(3 + |\nabla u|^2)\, u_{,\beta}\right]_{,\beta} = 0 \quad \text{on } D\,. \tag{23}$$

When (17), (18) and (21) hold, the components of nominal stress in (8) reduce further to

$$\sigma_{\alpha\beta} = 0\,, \quad \sigma_{\alpha 3} = \sigma_{3\alpha} = 2\, W_0'(3 + |\nabla u|^2)\, u_{,\alpha}\,, \quad \sigma_{33} = 0 \quad \text{on } D\,. \tag{24}$$

From (6), (2) and (24), one can determine the true stresses τ_{ij} :

$$\tau_{\alpha\beta} = 0, \quad \tau_{\alpha 3} = \tau_{3\alpha} = 2\, W_0'(3 + |\nabla u|^2)\, u_{,\alpha}, \quad \tau_{33} = 2\, W_0'(3 + |\nabla u|^2)\, |\nabla u|^2\,. \tag{25}$$

Since $I_3 = 1$ and $(18)_{1,3}$ hold, the strain energy density in anti-plane shear for a material with an elastic potential of the form (17) is given by

$$W = W_0(3 + |\nabla u|^2)\,. \tag{26}$$

We note that (1), (23)-(26) are identical with the field equations for anti-plane shear of *incompressible* materials of the generalized Neo-Hookean type; see [3].

For homogeneous, isotropic hyperelastic materials characterized by elastic potentials of the form (17), the Lamé constants λ and μ associated with infinitesimal deformations from an unstressed reference state can be found by approximating W in (17) near the undeformed state and comparing the result with the conventional expression for W in terms of λ and μ delivered by classical linearized theory. This leads to

$$\mu = 2\,W_0'(3) = M(0)\,, \quad \lambda = 4\Big\{\Big[f''(1) - 1\Big]W_0'(3) - W_0''(3) + g''(1)\Big\}. \qquad (27)$$

It is natural to require that the displacement equations of equilibrium be strongly elliptic at the reference state. Since this will be true if and only if $\mu > 0$ and $\lambda + 2\mu > 0$ (see [6]), we impose the following additional requirements on W_0, f and g:

$$W_0'(3) > 0\,, \qquad g''(1) + W_0'(3)f''(1) - W_0''(3) > 0\,. \qquad (28)$$

Consider an *arbitrary* deformation of a body composed of a compressible hyperelastic material with elastic potential $W(I_1, I_2, I_3)$. If $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches of the deformation, one has

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2\,. \qquad (29)$$

Let

$$W(I_1, I_2, I_3) = W^*(\lambda_1, \lambda_2, \lambda_3)\,. \qquad (30)$$

It follows easily from (7), (30) and (29) that the principal true stresses τ_1, τ_2, τ_3 are given by

$$\tau_i = (\lambda_i / J)\, \partial W^*(\lambda_1, \lambda_2, \lambda_3)/\partial\lambda_i\,, \qquad \text{no sum on } i\,. \qquad (31)$$

The ***Baker-Ericksen inequalities*** assert that the greater of any two principal stresses corresponds to the greater of the associated pair of principal stretches when the stretches are distinct; thus:

$$(\tau_i - \tau_j)(\lambda_i - \lambda_j) > 0 \quad \text{if } \lambda_i \neq \lambda_j \ , \ i,j = 1,2,3 . \tag{32}$$

For the special class of materials whose elastic potentials are of the form (17), one finds from (17), (29)-(31) that the Baker-Ericksen inequalities (32) hold if and only if

$$W_o'(I_1) + W_o''(I_1)\, f(I_3) > 0 \quad \text{for } 0 < I_3 < (I_1/3)^3 \ . \tag{33}$$

In [2], a necessary and sufficient condition on an isotropic elastic potential for the existence of non-trivial states of finite anti-plane shear was established under the hypothesis that the axial equilibrium equation (13) should remain elliptic for all out-of-plane displacement fields u. The condition given in (2) remains sufficient without this hypothesis, and it is indeed satisfied for all elastic potentials of the form (17).

4. An example. Among the materials characterized by elastic potentials of the form (17), (18), (21) and (28), there is a subclass that corresponds to the *Neo-Hookean* incompressible material. For this subclass, one has $W_o(I_1) = \mu/2\,(I_1 - 3)$, where $\mu > 0$ is the shear modulus for infinitesimal deformations, so that (17) becomes

$$W(I_1, I_3) = \mu/2\,(I_1 - 3) + h(J) , \tag{34}$$

where we have written

$$h(J) = \mu/2\, f(I_3) + g(I_3) \ , \ I_3 = J^2 . \tag{35}$$

For all materials in this special subclass, the differential equation (23) of anti-plane shear

reduces to Laplace's equation, and the associated true stresses of (25) become

$$\tau_{\alpha\beta} = 0 , \quad \tau_{\alpha 3} = \tau_{3\alpha} = \mu\, u_{,\alpha} , \quad \tau_{33} = \mu\, |\nabla u|^2 , \tag{36}$$

just as in the case of the Neo-Hookean material [3,5]. As in the latter case, the theory of finite anti-plane shear for compressible materials characterized by elastic potentials of the form (34) is indistinguishable from that for infinitesimal anti-plane deformations *except* for the presence of the axial normal stress τ_{33}, a nonlinear effect exploited in [5].

For elastic potentials of the form (34), the Baker-Ericksen inequality (33) reduces to the requirement $\mu > 0$.

One can show that the full three-dimensional displacement equations of equilibrium associated with the potential (34) are strongly elliptic at a given deformation if and only if

$$\mu > 0 , \quad J^2 h''(J) + \mu \lambda_i^2 > 0 , \quad i = 1,2,3 , \tag{37}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches and $J = \lambda_1\lambda_2\lambda_3$ the Jacobian of the given deformation. Since the smallest principal stretch made be made arbitrarily small while keeping J fixed at any specified positive value, it follows from (37) that the three-dimensional displacement equations of equilibrium will be strongly elliptic at *all* deformations if and only if

$$\mu > 0 , \quad h''(J) \geq 0 \quad \text{for all } J > 0 . \tag{38}$$

Elastic potentials of the form (34) correspond to special cases of the Hadamard materials studied extensively by John [7].

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